SHARP DISTORTION THEOREMS ASSOCIATED WITH THE SCHWARZIAN DERIVATIVE

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1. Introduction and Results

Let f be analytic in the unit disk and let $Sf = (f''/f')' - (1/2)(f''/f')^2$ be its Schwarzian derivative. If f is locally univalent and satisfies

$$|Sf(z)| \le \frac{2}{(1-|z|^2)^2} \tag{1.1}$$

then f is univalent. Furthermore, if the stronger inequality

$$|Sf(z)| \le \frac{2t}{(1-|z|^2)^2} \tag{1.2}$$

holds for some $0 \le t < 1$, then f has a quasiconformal extension to the plane. On the other hand, if f is univalent in the first place then

$$|Sf(z)| \le \frac{6}{(1-|z|^2)^2}. \tag{1.3}$$

These are well known and important theorems of Nehari, Ahlfors and Weill, and Krauss. We refer to Lehto's book [8] for a discussion of these results and for the properties of the Schwarzian that we shall need. The constants 2 in (1.1) and 6 in (1.3) are sharp. An example for the latter is the Koebe function $k(z) = z(1-z)^{-2}$ which has Schwarzian $Sk(z) = -6/(1-z^2)^2$. We also remark that the class of univalent functions satisfying (1.1) is quite large; for instance, it contains the class of convex mappings.

Since $S(M \circ f) = Sf$ for any Möbius transformation M, the inequalities above are independent of any such normalization $M \circ f$ of f, and this is an interesting feature of these results. However, if we require the normalization $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$ then we can obtain, rather simply, sharp and explicit upper and lower bounds on $|f|$ and $|f'|$ for functions which satisfy (1.1) or (1.2) . We introduce the following functions. Let

$$n(z) = \frac{1}{\sqrt{2}} \frac{(1+z)^{\sqrt{2}} - (1-z)^{\sqrt{2}}}{(1+z)^{\sqrt{2}} + (1-z)^{\sqrt{2}}}, \quad N(z) = \frac{1}{2} \log \frac{1+z}{1-z}, \tag{1.4}$$

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and, for $0 \le t < 1$, let

$$A(z;t) = A_t(z) = \frac{1}{\sqrt{1-t}} \frac{(1+z)^{\sqrt{1-t}} - (1-z)^{\sqrt{1-t}}}{(1+z)^{\sqrt{1-t}} + (1-z)^{\sqrt{1-t}}}. \tag{1.5}$$

These functions are analytic for $|z| < 1$ and are normalized as above. Their Schwarzian derivatives are

$$Sn(z) = \frac{-2}{(1-z^2)^2}, \quad SN(z) = \frac{2}{(1-z^2)^2}, \tag{1.6}$$

and

$$SA_t(z) = \frac{2t}{(1-z^2)^2}. \tag{1.7}$$

For the function $A(z;-t)$, which we also need, one has $SA_{-t}(z) = -SA_t(z)$.

The mapping properties of these functions are easy enough to see. The function $N(z)$, of course, maps the disk onto the horizontal strip $|Im(\zeta)| < \pi/2$. The other functions have to do with the pair of circles

$$|\zeta \pm \frac{i}{\alpha} \cot \frac{\pi}{\alpha}| = \frac{1}{\alpha} \frac{1}{\sin \frac{\pi}{\alpha}}, \quad 0 < \alpha < 2.$$

The function $n(z)$ maps the disk onto the region interior to the union of the two circles with $\alpha = \sqrt{2}$, while the image of $A(z;t)$ is the region bounded by the intersection of the two circles with $\alpha = \sqrt{1-t}$. The function $A(z;-t)$ is like $n(z)$ with $\alpha = \sqrt{1+t}$.

These functions are extremal in the class of normalized functions which satisfy (1.1) or (1.2).

Theorem 1: *Let f be analytic in the unit disk with $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$.*

(i) If f satisfies (1.1) then

$$n(|z|) \le |f(z)| \le N(|z|) \tag{1.8}$$

$$n'(|z|) \le |f'(z)| \le N'(|z|)\,. \tag{1.9}$$

(ii) If f satisfies (1.2) then

$$A(|z|;-t) \le |f(z)| \le A(|z|;t) \tag{1.10}$$

$$A'(|z|;-t) \le |f'(z)| \le A'(|z|;t)\,, \tag{1.11}$$

where A' means differentiation with respect to z. In any of these inequalities, if equality holds at one point other than the origin then f is conjugate by a rotation to one of $n(z), N(z), A(z;t)$, or $A(z;-t)$ as the case may be.

It is interesting to note that in the hypothesis of Theorem 1 we really only have to assume that the function is locally univalent and meromorphic. For the classes of functions we study, the absence of poles is actually a consequence of the normalization. This will become apparent from the proof of Corollary 2, below.

The functions $A(z;\pm t)$ give a 1-parameter family from $n(z) = A(z;-1)$ to $N(z) = \lim_{t\to 1} A(z;t)$ with $A(z;0) = z$. Notice also that normalized functions satisfying the Ahlfors–Weill condition are bounded. This turns out to be quite useful in other contexts, [2].

The simple character of the bounds in Theorem 1 makes it possible to deduce fairly directly several other interesting facts. For example, a normalized function satisfying the Ahlfors-Weill condition (1.2) has a Hölder continuous extension to the closed disk, with a better exponent than that which one gets just from the quasiconformal extension.

Corollary 1: *If f is analytic in the unit disk, is normalized as in Theorem 1, and satisfies (1.2) then f has a Hölder continuous extension to $|z| \le 1$ with*

$$|f(z_1) - f(z_2)| \le \frac{4\pi}{\sqrt{1-t}} |z_1 - z_2|^{\sqrt{1-t}},$$

for all z_1, z_2 in $|z| \le 1$. The exponent $\sqrt{1-t}$ is sharp.

In their paper [7], Gehring and Pommerenke obtain information on extensions to the closed disk of functions which satisfy Nehari's condition (1.1). Their precise estimates do not follow directly from the first part of Theorem 1.

Finally, a number of authors have found the normalization $f''(0) = 0$ to be helpful in studying the Schwarzian, see especially [7]. How restictive is it, if one wants to stick with analytic rather than meromorphic functions? This points up a curious fact about (locally) univalent functions. If $f(z) = z + a_2 z^2 + \ldots$ is analytic in the disk then $g = f/(1 + a_2 f)$ again has $g(0) = 0, g'(0) = 1$ and also $g''(0) = 0$, and has the same Schwarzian as f. It will be analytic in the disk if f does not assume the value $-{a_2}^{-1}$ there. This is actually the case for functions which satisfy the Nehari condition (1.1). It is not true for the full class of univalent functions, though it is closer to being true than one might think. We have the following.

Corollary 2: *Let $f = z + a_2 z^2 + \ldots$ be analytic and locally univalent in the unit disk.*

(i) If f satisfies (1.1) then f does not assume the value $-{a_2}^{-1}$ in $|z| < 1$.
(ii) If f satisfies

$$|Sf(z)| \le \frac{2\alpha}{(1 - |z|^2)^2}, \tag{1.12}$$

where $1 < \alpha$, then f does not assume the value $-{a_2}^{-1}$ in the disk $|z| < \tanh\left(\frac{\pi}{2\sqrt{\alpha-1}}\right)$.

To say more about the second part of this theorem, we let $|z| < r_0(\alpha)$ be the largest disk on which any f satisfying (1.12) does not assume the value $-{a_2}^{-1}$. Then $1 \ge r_0(\alpha) \ge \tanh(\pi/(2\sqrt{\alpha - 1}))$, but we do not know whether the lower bound is sharp for any α, even for univalent functions. We do know that for $\alpha = 3$, which includes the full class of univalent functions, one has $.804\ldots =$

$\tanh(\pi/(2\sqrt{2})) \leq r_0(3) \leq \sqrt{3}/2 = .866\ldots$. Examples are given by the mappings

$$f(z) = \frac{z - z^2\cos\phi}{(1 - e^{i\phi}z)^2} = z + (2e^{i\phi} - \cos\phi)z^2 + \ldots.$$

For $\cos\phi \neq 0, 1$ this function maps the disk onto the complement of a non-radial halfline with finite endpoint at $i/(4\sin\phi)$ and inclination $(3\pi/2) - 2\phi$, see [3, **?**, **?**]. For the Schwarzian we compute

$$Sf(z) = \frac{6\sin^2\phi}{(e^{i\phi} - z)^2(e^{-i\phi} - z)^2},$$

and then that $(1 - |z|^2)^2|Sf(z)| \leq 6$ with equality holding along the hyperbolic geodesic joining $e^{i\phi}$ and $e^{-i\phi}$ in the disk. The most convincing way of seeing this, if perhaps not the most direct, is to observe that it is possible to write $f(z) = T_1(k(T_2(z)))$ where T_1 is a similarity, k is the Koebe function, and T_2 is a Möbius transformation of the disk onto itself. The transformation formula for the Schwarzian and the invariance of the Poincaré metric then give the result. It is easy to solve $f(z) = -(2e^{i\phi} - \cos\phi)^{-1}$ and one finds a real solution varying between $\sqrt{3}/2$ and 1 when $\cos\phi > 0$ (and between $-\sqrt{3}/2$ and -1 when $\cos\phi < 0$). The minimum $\sqrt{3}/2$ occurs when $\cos\phi = 1/\sqrt{3}$.

The proofs of the Theorem and its Corollaries will be given in Section 2. In Section 3 we show how these results can be applied to get a classical coefficient bound. In Section 4 we remark briefly on the situation for another univalence criterion.

2. Proofs

The main result is Theorem 1, giving bounds on f and f'. The arguments use only basic comparison theorems for differential equations. The normalizations $f'(0) = 1, f''(0) = 0$ give particularly convenient initial conditions. In this connection we call attention to the interesting paper of Essén and Keogh [4]. These authors employ similar techniques, but with different aims. We use some of their lemmas, but it does not seem that the results of the present paper can be deduced directly from their theorems.

The starting point is the fact that for a function f as in Theorem 1 one has the representation

$$f(z) = \int_0^z u^{-2}(\zeta)\, d\zeta, \tag{2.1}$$

where u satisfies the the initiial value problem

$$u'' + \frac{1}{2}Sf\, u = 0, \quad u(0) = 1, u'(0) = 0. \tag{2.2}$$

The following Lemmas will apply directly to the proof of the second part of Theorem 1, but the same reasoning can be used with obvious changes for the proof of the first part. For convenience, we will write $q(z) = (1/2)Sf(z)$ and $p(r) = t(1-r^2)^{-2}$ for $0 \le r < 1$. The hypothesis of the second part of Theorem 1 is then that $|q(z)| \le p(|z|)$.

Lemma 2.1. *Let $w(x)$ defined on $[0,1)$ be real valued and satisfy*

$$w'' + p\,w \ge 0, \quad w(0) = 1, w'(0) = 0.$$

Then $w \ge h$ where

$$h'' + p\,h = 0, \quad h(0) = 1, h'(0) = 0. \tag{2.3}$$

Proof. Let $y = w - h$. Then $y'' + p\,y \ge 0, y(0) = 0,\ y'(0) = 0$. The solutuion g to $g'' + p\,g = 0$ with $g(0) = 0, g'(0) = 1$ is bounded and positive on $(0,1)$ with $g(1) = 0$, see [6, **?**]. Then for $\varepsilon > 0$, we have $(y + \varepsilon\, g)'' + p\,(y + \varepsilon\, g) \ge 0$, with $(y + \varepsilon\, g)(0) = 0$ and $(y + \varepsilon\, g)'(0) > 0$. A standard Sturm comparison theorem guarantees that $y + \varepsilon\, g \ge 0$ on $[0,1]$. Since $\varepsilon > 0$ is arbitrary and g is bounded, the Lemma follows. □

Lemma 2.2. *The solution u of (2.2) satisfies $|u|'' + p\,|u| \ge 0$ on $[0,1)$.*

Proof. Let $\eta = |u|^2 = u\,\bar{u}$. Since $u = (f')^{-\frac{1}{2}} \ne 0$, we know that $|u|$ is differentiable. Hence $2|u|\,|u|' = \eta' = u'\,\bar{u} + u\,\bar{u}'$, and then

$$\eta'' = u''\bar{u} + u\bar{u}'' + 2|u'|^2 = 2|u|\,|u|'' + 2(|u|')^2 = -(q+\bar{q})|u|^2 + 2|u'|^2 \ge -2p|u|^2 + 2|u'|^2.$$

But $|u'|^2 \ge (|u|')^2$ and $|u| > 0$, so we conclude that $|u|'' \ge -p|u|$ as desired. □

The argument in this proof can be applied along any segment $[0, e^{i\theta})$. It follows from Lemmas 1 and 2 that

$$|u(z)| \ge h(|z|) \quad \text{for all } z \text{ with } |z| < 1. \tag{2.4}$$

The solution h to (2.3) is classical and is given by

$$h(z) = \frac{1}{2}\sqrt{1-z^2}\left\{\left(\frac{1+z}{1-z}\right)^{\gamma} + \left(\frac{1+z}{1-z}\right)^{-\gamma}\right\}, \tag{2.5}$$

where $\gamma = \frac{1}{2}\sqrt{1-t}$, see [6, **?**].

We now put

$$A_t(z) = \int_0^z h^{-2}(\zeta)\,d\zeta.$$

Surprisingly, the integration is quite simple and yields the expression in (1.5). The upper bounds for $|f|$ and $|f'|$ in (1.10) and (1.11) follow at once from (2.4).

In order to establish the corresponding lower bounds we consider the solution v to

$$v'' - p\,v = 0, \quad v(0) = 1, v'(0) = 0. \tag{2.6}$$

It follows from Lemma 8 in [4] that $|u(z)| \le v(|z|)$ for all z with $|z| < 1$. Again, v can be found explicitly to be,

$$v(z) = \frac{1}{2}\sqrt{1-z^2}\left\{\left(\frac{1+z}{1-z}\right)^{\delta} + \left(\frac{1+z}{1-z}\right)^{-\delta}\right\},$$

where $\delta = \frac{1}{2}\sqrt{1+t}$. Then

$$\int_0^z v^{-2}(\zeta)\,d\zeta$$

gives the expression for $A(z;-t)$ and the lower bound for $|f'|$ in (1.11). We employ a standard argument to derive the lower bound for $|f|$ in (1.10). Suppose that on the circle $|z| = r < 1$ the minimum of $|f|$ is assumed at the point z_1. The inverse of f is then defined all along the segment $I = [0, \zeta_1]$, $\zeta_1 = f(z_1)$. Thus with $J = f^{-1}(I)$ we have

$$|\zeta_1| = \int_I |d\zeta| = \int_J |f'(z)|\,|dz| \ge \int_J A'_{-t}(|z|)\,|dz| \ge \int_0^{|z_1|} A'_{-t}(|z|)\,|dz| = A_{-t}(|z_1|), \tag{2.7}$$

where the final integral is along the segment form 0 to z_1.

Next we consider the cases of equality:

(*i*) $|f(z_0)| = A_t(|z_0|)$,
(*ii*) $|f'(z_0)| = A'_t(|z_0|)$,
(*iii*) $|f(z_0)| = A_{-t}(|z_0|)$,
(*iv*) $|f'(z_0)| = A'_{-t}(|z_0|)$,
for some $z_0 \ne 0$. By taking $e^{-i\theta}f(e^{i\theta}z), e^{i\theta} = z_0/|z_0|$, we may assume that $z_0 = x_0 > 0$. Case (i) will be reduced to case (ii), and case (iii) will be reduced to case (iv).

Suppose that $|f'(x_0)| = A'_t(x_0)$. Then the function $y = |f'|^{-\frac{1}{2}} - (A'_t)^{-\frac{1}{2}}$ satisfies

$$y'' + p\,y \ge 0, \quad y \ge 0,\ y'(0) = 0$$

and $y(x_0) = 0$. It follows from Lemma 4 in [4] that y is identically zero, that is, that $|f'(x)| = A'_t(x)$ for all x in $[0, x_0]$, and therefore on all of $[0, 1)$ since both sides are analytic. We will show that $(2q(x) =)\, Sf(x) = SA_t(x)\, (= 2p(x))$. Along $[0, 1)$ let $\psi = |f'|^{-1} = |u|^2 = (A'_t)^{-1} = v^2$. Then, as in Lemma 2.2,

$$\psi'' = -(q+\bar{q})|u|^2 + 2|u'|^2 = -2pv^2 + 2(v')^2.$$

But $|u'|^2 \ge (|u|')^2 = (v')^2$ and $-(q+\bar{q}) \ge -2p$. Thus we must have equality in both cases, which implies that $q = p$ on $[0, 1)$ and hence in the entire disk. Therefore $f = A_t$ because of the normalizations.

This settles case (ii). To get case (i) from this, suppose that $|f(x_0)| = A_t(x_0)$. Then the string of inequalities,

$$|f(x_0)| \leq \int_0^{x_0} |f'(x)|\,dx \leq \int_0^{x_0} A_t'(x)\,dx = A_t(x_0),$$

implies that $|f'(x)| = A_t'(x)$ for all x in $[0, x_0]$ and we are back to case (ii).

We now reduce case (iii) to case (iv). If $|f(x_0)| = A_{-t}(x_0)$ then in the inequalities in (2.7) we have equality everywhere. This implies that J is the interval $[0, x_0]$ and that $|f'(x)| = A_{-t}^{'}(x)$ along the interval. The fact that $f(z) = A_{-t}(z)$ for all $|z| < 1$ is then a consequence of Lemma 8 in [4], where the authors characterize the case of equality at a point $x_0 \neq 0$ of the functions $|u|$ and $v = (A_{-t}^{'})^{-2}$. Equality at a single nonzero point implies equality everywhere. This finishes the proof of the second part of Theorem 1. As we mentioned above, the proof for the first half runs along the same lines, using the explicit solutions to the differential equation for the normalized functions satisfying $Sf = \pm 2/(1-z^2)^2$.

Suppose now that f is normalized as above with $(1-|z|^2)^2|Sf(z)| \leq 2t$. The proof that f satisfies a Hölder condition, as asserted in Corollary 1, is a straightforward consequence of the upper bound (1.11). From (1.11) we have

$$|f'(z)| \leq 4\frac{(1+|z|)^{2\nu-1}(1-|z|)^{2\nu-1}}{((1+|z|)^{2\nu}+(1-|z|)^{2\nu})^2},$$

where $2\nu = \sqrt{1-t}$. It follows that

$$|f'(z)| \leq \frac{4^{1-2\nu}}{(1-|z|)^{1-2\nu}}.$$

From here we follow the argument as in [7], which we give for the convenience of the reader. For z_1, z_2 in the disk, let Γ be the hyperbolic segment joining z_1 and z_2. Then Γ has euclidean length $l \leq \frac{\pi}{2}|z_1 - z_2|$ and $\min(s, l-s) \leq \frac{\pi}{2}(1-|\zeta|)$ for each ζ on Γ, where s is the euclidean arclength of the part of Γ between z_1 and ζ. Thus

$$|f(z_1)-f(z_2)| \leq \int\limits_\Gamma |f'(\zeta)|\,|d\zeta| \leq 4^{1-2\nu}\int\limits_\Gamma \frac{|d\zeta|}{(1-|\zeta|)^{1-2\nu}} \leq 2\cdot 4^{1-2\nu}\int_0^{l/2} (\frac{\pi}{2})^{1-2\nu}\frac{ds}{s^{1-2\nu}}.$$

Integration, together with $l \leq \frac{\pi}{2}|z_1 - z_2|$, yields

$$|f(z_1) - f(z_2)| \leq \frac{8^{1-2\nu}}{4\nu}\pi|z_1 - z_2|^{2\nu} \leq \frac{4\pi}{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}}.$$

This shows uniform continuity in the disk, and hence the extension of f to the closure satisfies the same Hölder condition. The example $A_t(z)$ shows that the exponent is sharp.

This completes the proof of Corollary 1. We have one additional minor comment on this. The bounds in (1.11) give

$$A(1-|z|)^{\mu-1} \leq |f'(z)| \leq B(1-|z|)^{\nu-1},$$

where $\mu = \sqrt{1+t}$ and $\nu = \sqrt{1-t}$, as above. The function $g = f^{-1}$ is defined in the image Ω of the disk under f, which is a quasidisk. Hence, by a result in [13], g will be Hölder continuous with exponent η if and only if $|g'(\zeta)| \leq C\,\mathrm{dist}(\zeta, \partial\Omega)^{\eta-1}$, where dist denotes the euclidean distance. Using both the upper and lower bounds for $|f'|$ above, one can show easily that $|g'(\zeta)| \leq C\,\mathrm{dist}(\zeta, \partial\Omega)^{(1-\mu)/\nu}$. The exponent $(1-\mu)/\nu$ is of the form $\eta - 1$ for $\eta > 0$ if and only if $t < \sqrt{3}/2$. Thus for $0 \leq t < \sqrt{3}/2$ the function g is Hölder continuous with the unattractive exponent $\eta = (1+\sqrt{1-t}-\sqrt{1+t})/\sqrt{1-t}$. It is known, however, that a conformal map from a quasidisk onto the disk is Hölder continuous with some exponent $> (1/2)$, see [9],[10],[1]. This value of η improves these results, in this case, for small t, and $\eta \to 1$ as $t \to 0$, which is what one expects.

Next, we consider the problem of normalizing $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ to $g = f/(1+a_2f) = z + (a_3 - {a_2}^2)z^3 + \cdots$ while still staying within the class of analytic functions. The function g is meromorphic in $|z| < 1$ and the issue is whether it has a pole, at a point z_0 where $f(z_0) = -{a_2}^{-1}$. Suppose first that f satisfies Nehari's condition (1.1). Then so does g, where the Schwarzian at a pole is defined via inversion. If, say, $|z| < r \leq 1$ is the largest disk on which g is analytic, then the same arguments used in the proof of Theorem 1 show that

$$|g(z)| \leq \frac{1}{2}\log\frac{1+|z|}{1-|z|} \tag{2.8}$$

in $|z| < r$. Then g is bounded on $|z| < r$ and hence cannot have a pole in $|z| < 1$. Therefore f does not assume the value $-{a_2}^{-1}$.

Next suppose that f, hence g, satisfies $(1-|z|^2)^2|Sf(z)| \leq 2\alpha$ for $\alpha > 1$. Again suppose that the largest disk on which g is analytic is $|z| < r$. The function

$$F_\alpha(z) = \frac{1}{\sqrt{\alpha-1}}\tan\left\{\frac{\sqrt{\alpha-1}}{2}\log\frac{1+z}{1-z}\right\}$$

has $F_\alpha(0) = 0, F_\alpha'(0) = 1, F_\alpha''(0) = 0$, with $SF_\alpha(z) = 2\alpha/(1-z^2)^2$, and F_α is analytic on the disk $|z| < r_1 = \tanh(\pi/(2\sqrt{\alpha-1}))$. Similar differential equations arguments show that $|g(z)| \leq F_\alpha(|z|)$ on the disk $|z| < \min\{r, r_1\}$. Hence we must have $r \geq r_1$, i.e. f cannot assume the value $-{a_2}^{-1}$ in the disk $|z| < r_1$, which is what we wanted to show.

Finally, we note that the normalization and the lower bound in (1.8) can be used to give a quick proof of Nehari's original univalence criterion. Suppose that f is analytic, satisfies (1.1), and that $f(z_1) = f(z_2)$ for two points in the disk. The condition (1.1) is invariant under compositions $f \circ M$ with a self-mapping of the disk, and we may therefore suppose that $z_1 = 0$. Now normalize f so that

$f(0) = 0, f'(0) = 1, f''(0) = 0$ by a Möbius transformation of the range. Then the normalized f is again analytic, and the lower bound in (1.8) implies that $z_2 = z_1 = 0$.

3. Covering Properties and Bounds on a_2

If $f(z) = z + a_2 z^2 + \cdots$ is analytic and (locally) univalent in the disk, then bounds on Sf of the type we have been considering lead to covering properties of f and to bounds on a_2. This was one of the main points of the Essén and Keogh paper. It was also investigated in a paper by Farkas [5] who in turn recoverd some results of Pommerenke [12]. Here, we want to show how the further normalization $f''(0) = 0$ can lead back, in a new way, to the Big Bang of univalent functions: $|a_2| \leq 2$. This is only for people who do not like Faber's square root trick. One can do more along these lines, but this is a typical application of the methods of this paper.

First suppose that a normalized function $f(z) = z + b_3 z^3 + \cdots$ is analytic and univalent in the disk. Then (2.1) and (2.2) hold as before. Since $|Sf(z)| \leq 6(1-|z|^2)^{-2}$ it follows as in earlier arguments that $|f(z)| \geq \varphi(|z|)$, and $|f'(z)| \geq \varphi'(|z|)$, where

$$\varphi(z) = \int_0^z v^{-2}(\zeta)\, d\zeta,$$

with

$$v'' - 3(1-z^2)^{-2} v = 0, \quad v(0) = 1, v'(0) = 0.$$

The solution of the differential equation is $v(z) = (1-z^2)^{-\frac{1}{2}}(1+z^2)$ which yields $\varphi(z) = z/(1+z^2)$. Hence any such normalized univalent function satisfies

$$|f(z)| \geq \frac{|z|}{1+|z|^2} \tag{3.1}$$

$$|f'(z)| \geq \frac{1-|z|^2}{(1+|z|^2)^2}\,. \tag{3.2}$$

Equality at $z_0 \neq 0$ in either inequality implies that $f(z) = z/(1 + e^{-2i\theta} z^2)$, $e^{i\theta} = z_0/|z_0|$. In particular, we conclude that the range of f covers a disk centered at the origin of radius at least $1/2$.

Now let $g(z) = z + a_2 z^2 + \cdots$ be analytic and univalent in $B = \{|z| < 1\}$. Then $f = g/(1 + a_2 g)$ is normalized as above, with a possible pole at a (unique) point where $g(z) = -{a_2}^{-1}$. We claim that even in this case the range of f covers at least $|w| < \frac{1}{2}$. If g never assumes $-{a_2}^{-1}$ then this follows from the previous analysis. Suppose $g(z_0) = -{a_2}^{-1}, z_0 \neq 0$. Let $B' = B \backslash \{z_0\}$, and let σ be the shorter radial slit from z_0 to ∂B. Then f is regular in the simply connected domain $B \backslash \sigma$ and the representation (2.1) holds there. It follows as before that $|f'(z)| \geq \varphi'(|z|)$ for $z \,\epsilon\, B \backslash \sigma$, and by continuity this is also true in B'.

By considering $r^{-1}f(rz)$ and letting $r \uparrow 1$ we may assume that f is one-to-one on the unit circle. Let $\Gamma = f(\partial B)$ and let $\zeta_1 \,\epsilon\, \Gamma$ be such that $|\zeta_1| = \inf |\zeta|$ on Γ. Then I, the segment from 0 to ζ_1, lies in $f(\overline{B})$. With $J = f^{-1}(I)$ we have

$$|\zeta_1| = \int_I |d\zeta| = \int_J |f'(z)|\,|dz| \geq \int_J \varphi'(|z|)\,|dz| \geq \int_0^1 \varphi'(|z|)\,|dz| = \frac{1}{2},$$

where the last integral is along a radius from 0 to the circle. (Note that z_0 does not lie on J.)

This establishes the claim that the range of f does cover the disk $|w| < \frac{1}{2}$. Now, since $g = f/(1 - a_2 f)$ is analytic in $|z| < 1$ we must have $|a_2|^{-1} \geq \frac{1}{2}$, or $|a_2| \leq 2$. Finally, if $|a_2| = 2$ then $|\zeta_1| = \frac{1}{2}$ and it follows easily from the cases of equality in (3.2) that f is a Möbius transformation of the Koebe function and that g is a rotation of the Koebe function.

4. Other Univalece Criteria

Nehari also proved that

$$|Sf(z)| \leq \pi^2/2\,, \tag{4.1}$$

is a sufficient condition for a function to be univalent in the disk (see also [11]). According to [7] the stronger condition

$$|Sf(z)| \leq t\,\frac{\pi^2}{2}\,, \tag{4.2}$$

for some $0 \leq t < 1$, then implies that f has a quasiconformal extension to the plane. There are analogues of Theorem 1 and Corollaries 1 and 2 for this condition. The differential equations for the extremal functions are even easier to solve than before, and the comparison theorems show that a function satisfying (4.1) or (4.2), and normalized by $f(0) = 0, f'(0) = 1, f''(0) = 0$, is subject to the sharp bounds

$$\frac{2}{\pi\sqrt{t}} \tanh\left(\frac{\pi\sqrt{t}}{2}|z|\right) \leq |f(z)| \leq \frac{2}{\pi\sqrt{t}} \tan\left(\frac{\pi\sqrt{t}}{2}|z|\right), \tag{4.3}$$

$$\cosh^{-2}\left(\frac{\pi\sqrt{t}}{2}|z|\right) \leq |f'(z)| \leq \cos^{-2}\left(\frac{\pi\sqrt{t}}{2}|z|\right). \tag{4.4}$$

As before, the cases of equality at a single $z \neq 0$ implies that f is conjugate by a rotation to the corresponding extremal function.

As for Corollary 1, it is interesting to note that a normalized function satisfying (4.2) for some $t < 1$ has an extension to the closed disk which is *Lipschitz*. The argument is as before, using (4.4). Finally, it follows as in the proof of Corollary 2 that a function $f(z) = z + a_2 z^2 + \cdots$ satisfying (4.1) does not assume the value $-{a_2}^{-1}$ in the unit disk, i.e, it is always possible to normalize further to get $f''(0) = 0$.

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